

THE DERIVATIVE OF INFLUENCE FUNCTION,
LOCATION BREAKDOWN POINT,
GROUP INFLUENCE
AND
REGRESSION RESIDUALS' PLOTS

Yannis G. Yatracos

Faculty of Communication and Media Studies

Cyprus University of Technology

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Summary

In several linear regression data sets, $Y(\in R)$ on $\mathbf{X}(\in R^p)$, *visual comparisons* of L_1 and L_2 -residuals' plots indicate bad leverage cases. The phenomenon is confirmed theoretically by introducing Location Breakdown Point (*LBP*) of a functional T : any point where the derivative of T 's Influence Function either takes values at infinities or does not exist. Guidelines for the plots' visual comparisons as diagnostic are provided. The new tools used include E -matrix and suggest influence diagnostic *RINFIN* which measures the *distance* in the *derivatives of L_2 -residuals* at (\mathbf{x}, y) from model F and from gross-error model $F_{\epsilon, \mathbf{x}, y}$. The larger $RINFIN(\mathbf{x}, y)$ is, the larger (\mathbf{x}, y) 's influence in L_2 -regression residual is. *RINFIN* allows measuring *group influence* of k \mathbf{x} -*neighboring* data cases in a size n sample using their average, $(\bar{\mathbf{x}}_k, \bar{y}_k)$, as one case with weight $\epsilon = k/n$. For high dimensional, simulated data, the misclassification proportion of bad leverage cases in data's *RINFIN*-ordering decreases to zero as p increases, thus reconfirming the *blessing of high dimensionality* in the detection of remote clusters. The visual diagnostic and *RINFIN* are successful in applications and complement each other.

Some key words: Breakdown Point, Influence Function, Least Absolute Deviation Residuals, Least Squares Residuals, Leverage, Location Breakdown Point, Local-Shift-Sensitivity, Masking, Residual's Influence Index (*RINFIN*)

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1 Introduction

Tukey (1962, p.60) wrote: “Procedures of diagnosis, and procedures to extract indications rather than extract conclusions, will have to play a large part in the future of data analyses and graphical techniques offer great possibilities in both areas.”

In linear regression of Y on \mathbf{X} it is often assumed that the data follows probability model F ; $Y \in R, \mathbf{X} \in R^p$. However, “It also happens not infrequently that only part of the data obeys a different model.” (Hampel *et. al.*, 1986). Thus, in reality, data may follow gross-error model $F_{\epsilon,G} = (1 - \epsilon)F + \epsilon G$ (Huber, 1964); G is gross-error probability, $0 < \epsilon < 1$. The goal of this work is to provide *simple and fast procedures* for extracting indications when remote cases (from G) affect the statistical analysis in least squares (L_2) regression. These procedures are particularly useful for Big Data, when the number of predictors, p , and the sample size, n , increase to infinity; for L_2 -regression $p < n$.

The initial motivation was provided by the observation, in several data sets, that: neighboring, remote factor space cases, (\mathbf{x}, y) , have least absolute deviation (L_1) regression residuals significantly larger in size than the corresponding L_2 -regression residuals; see, e.g., Figures 1 and 2 in section 4.

This phenomenon is theoretically confirmed herein using new tools: E -matrix and derivatives of the regression coefficients’ Influence Functions. The latter allow calculating changes in L_1 and L_2 -regressions residuals for small perturbations of (\mathbf{x}, y) from F and also from $F_{\epsilon,G}$. The calculation of L_1 -residuals’ changes is possible when (\mathbf{x}, y) is *not* L_1 -Location Breakdown Point (LBP), thus first order linear approximation of the L_1 -residual near (\mathbf{x}, y) is valid. LBP complements the notion of *Weight Breakdown Point* (Hampel, 1971). Derivatives of Influence Functions indicate a new influence diagnostic: *RINFIN* (see below).

A simple graphical method is thus proposed to detect rapidly in linear regression data remote cases, (\mathbf{x}, y) , affecting drastically L_2 -regression coefficients. Plots of absolute regression residuals against square \mathbf{x} -length provide the *visual indications* when L_1 -residuals’ sizes for \mathbf{x} -remote cases are larger, e.g. double in size, than the corresponding L_2 -residuals, thus causing a larger *visual gap* in the L_1 plot. A different pattern in the residuals is used

to identify other types of outlying cases near a *LBP*, as described in section 4.

The regression diagnostic, Residual's Influence Index (*RINFIN*) for (\mathbf{x}, y) , is also introduced that measures *the distance in the derivatives of L_2 -residuals* when (\mathbf{x}, y) follows either probability F (the model) or its gross-error mixture $F_{\epsilon, \mathbf{x}, y}$, i.e. $(1 - \epsilon)F + \epsilon\Delta_{\mathbf{x}, y}$, $0 < \epsilon < 1, \Delta_{\mathbf{u}}$ unit mass at \mathbf{u} . The larger $RINFIN(\mathbf{x}, y)$ is, the larger (\mathbf{x}, y) 's influence in the L_2 -residual is. For a group of remote *\mathbf{x} -neighboring* cases from gross-error probability G , with proportion ϵ in the data, their group average $(\bar{\mathbf{x}}, \bar{y})$ is used as one case from $F_{\epsilon, \bar{\mathbf{x}}, \bar{y}}$ to calculate the group's influence, $RINFIN(\bar{\mathbf{x}}, \bar{y})$, that depends also on ϵ . This is an advantage over other methods that use group deletion to determine influence and are exposed *i)* to masking from neighboring G cases that remain in the model, *ii)* to a combinatorial explosion due to the very large number of groups to exclude.

RINFIN is successful with several known data sets and in simulations, especially when the dimension p of the data is large. In simulations with normal mixtures and n fixed, the misclassification proportion of bad leverage cases in the *RINFIN* ordering of the data decreases to zero as p increases. The effect of increase in p -values is equivalent to larger standardized distance between the means of F and G in $F_{\epsilon, G}$. A similar phenomenon has been observed and confirmed theoretically for mixture densities in a Projection Pursuit cluster detection method (Yatracos, 2013) due to the “separation” of the mixtures' components, measured by their Hellinger's distance, as p increases.

In a nutshell, the justification for the visual phenomenon and the form of *RINFIN* are presented for simple linear regression:

i) For $\epsilon(> 0)$ small in $F_{\epsilon, \mathbf{x}, y}$ residuals are compared,

$$\frac{|r_{2, \mathbf{x}, y}(x, y) - r_2(x, y)|}{|r_{1, \mathbf{x}, y}(x, y) - r_1(x, y)|} \approx C|r_2(x, y)|, \quad (1)$$

r_m and $r_{m, \mathbf{x}, y}$ are L_m -residuals, respectively, for F -regression and $F_{\epsilon, \mathbf{x}, y}$ -regression, $m = 1, 2$; C is constant, “ \approx ” denotes approximation. When (x, y) is gross-error and for L_1 and L_2 F -regressions $r_1(x, y) \approx r_2(x, y)$, with $|r_2(x, y)| > 1$, from (1) it follows for $F_{\epsilon, \mathbf{x}, y}$ -regression that L_2 residual of (x, y) is reduced more than its L_1 residual, especially when $|x|$ is large (because then $|r_2(x, y)|$ is also large).

ii) L_2 -residual's influence index of (x, y) from gross-error model $F_{\epsilon, x, y}$ is

$$RINFIN(x, y) = \epsilon \cdot \frac{|2r_2(x, y)(x - EX) - \beta_{1, L_2}[(x - EX)^2 + Var(X)]|}{Var(X)}; \quad (2)$$

L_2 -residual (r_2), slope (β_{1, L_2}), mean (EX) and variance ($VarX$) are all under F .

LBP of a statistical functional T is motivated and introduced in section 2 using \mathbf{x} -perturbations of $F_{\epsilon, \mathbf{x}}$. LBP is a point where the directional or one of the partial derivatives of T 's Influence Function (Hampel, 1971, 1974) either take values at infinities or do not exist. *Local-shift-sensitivity* (Hampel, 1974) cannot replace the derivatives, as explained. In section 3, regression coefficients' Influence Functions and their derivatives, obtained via E -matrices, are used to show that: in $F_{\epsilon, \mathbf{x}, y}$ -regression, when remote \mathbf{x} -case becomes slightly more extreme without reaching L_1 LBP , the size of the corresponding L_2 -residual is drastically reduced whereas the L_1 -residual is reduced less.

The graphical method and $RINFIN$ are supported by applications and simulations in section 4. Instead of square \mathbf{x} -length on the plot's horizontal axis, \mathbf{x} -length can be used. For some data sets, plotting regression residuals rather than their absolute values may be more informative. However, for remote gross-error model with small variance, e.g. cases 1-10 in Hawkins-Bradru-Kass (1984) data, absolute residuals are informative.

Robust residual plots are accompanied with confidence ellipsoids. Otherwise, visual indications from distorted residuals lead to inaccuracies. L_1 and L_2 residuals and the square length of dependent variables are used herein because they do not cause *unknown* amount of distortion in *relative* visual distances. For example, in the Stackloss Data plot (Rousseeuw and van Zomeren, 1990, p. 636, Figure 3) relative sizes of the *absolute* standardized Least Median of Squares (LMS) residuals of cases 1, 3, 4 and 21 differ from those in the L_2 -absolute residuals in Figure 2 herein.

In multiple regression, with observations from F , a case (\mathbf{x}, y) with factor space component, \mathbf{x} , far away from the bulk of F 's factor space is called *leverage case* (Rousseeuw and Leroy, 1987, Huber, 1997). A "good" leverage case is either near or on the regression hyperplane determined by F . A "bad" leverage case forces the F -hyperplane to change drastically when \mathbf{x} becomes more remote. The suggested comparisons of L_1 and L_2 residuals' plots and data's $RINFIN$ values reveal "bad" leverage cases.

The abundance of high dimensional data sets from various fields, with p and n both large, created the need for new methods to detect outliers/influential cases affecting linear regression analysis. High dimensional influence measures have been proposed among others by Alphons *et al.* (2013), Zhao *et al.* (2013, 2016) and references there in. She and Owen (2011) identify influential cases using nonconvex penalized likelihood. Influence Function in outlier detection has been used by Campbell (1978) and Boente *et al.* (2002). The influence of observations in estimates' values has been also studied by several authors, among others by Cook (1977), Cook and Weisberg (1980), Ruppert and Carroll (1980), Carroll and Ruppert (1985), Hampel (1985), Hampel *et al.* (1986), Ronchetti (1987), Rousseeuw and van Zomeren (1990), Ellis and Morgenthaler (1992), Bradu (1997), Flores (2015) and Genton and Hall (2016).

In Genton and Ruiz-Gazen (2010) an *observation* is influential “whenever a change in its value leads to a radical change in the estimate” and the *hair-plot* is introduced to identify it. Two influence measures are proposed using partial derivative of the *estimate*: *a)* the local, with a small perturbation in one coordinate of the observation, and *b)* the global, using the most extreme contamination for each coordinate. Differences in our work include: *i)* leverage cases affecting drastically L_2 -regression residuals are visually identified *combining information* from L_1 and L_2 residuals' plots, *ii)* the derivative of the estimate's influence function is used instead of the estimate's derivative, *iii)* *RINFIN* measures distance in residuals' derivatives and can be used to evaluate group influence of neighboring cases.

Work has been done to identify “bad” leverage cases using L_1 residuals. Barrodale (1968) compared L_1 and L_2 residuals for regression function $\sum_{j=1}^p a_j \phi_j(\mathbf{x})$ using tables for different \mathbf{x} -values; $\mathbf{x} \in R^d$, ϕ_j is known, a_j is an unknown coefficient, $j = 1, \dots, p$. Barnett and Lewis (1984) present the absolute residuals as a tool in outlier detection. Narula and Wellington (1985) looked for observations that do and do not affect the analysis in L_1 -regression using the residuals. Ellis and Morgenthaler (1992) and Bradu (1997) examined the performance of the L_1 regression estimator facing outliers in the *response* variable. Additional results on L_1 -regression and outliers may be found in Dodge(1987).

Recent results combine also information from L_1 and L_2 regression. Giloni and Padberg

(2002) presented a lower bound on total sum of absolute L_1 -residuals using the total sum of squared L_2 -residuals. Flores (2015) studied for a *particular* regression model the behavior of L_1 -estimates by comparing them with L_2 -estimates, and introduced *leverage constants* for a design matrix to determine whether leverage cases are good or bad.

Proofs are in the Appendix.

2 Location Breakdown Point (*LBP*)

Hampel (1971) introduced the influence function, $IF(\mathbf{x}; T, F)$, of a functional T at probability F ,

$$IF(\mathbf{x}; T, F) = \lim_{\epsilon \rightarrow 0} \frac{T[(1 - \epsilon)F + \epsilon\Delta_{\mathbf{x}}] - T(F)}{\epsilon}, \quad (3)$$

when this limit exists; $\mathbf{x}(\in R^p)$, $\Delta_{\mathbf{x}}$ is the probability distribution that puts all its mass at the point \mathbf{x} , $0 < \epsilon < 1$.

$IF(\mathbf{x}; T, F)$ determines the “bias” in the value of T at F due to an ϵ -perturbation of F with $\Delta_{\mathbf{x}}$:

$$T[(1 - \epsilon)F + \epsilon\Delta_{\mathbf{x}}] - T(F) \approx \epsilon IF(\mathbf{x}; T, F). \quad (4)$$

Definition 2.1 (*Hampel, 1971*) *The weight breakdown point is the upper bound on ϵ for which linear approximation (4) can be used.*

Discussing further concepts related to the influence function, Hampel (1974, p. 389) introduced *local-shift-sensitivity*,

$$\lambda^* = \sup_{\mathbf{x} \neq \mathbf{y}} \frac{|IF(\mathbf{x}; T, F) - IF(\mathbf{y}; T, F)|}{\|\mathbf{x} - \mathbf{y}\|}, \quad (5)$$

as “a measure for the *worst* (approximate) effect of wiggling the observations”; $\|\cdot\|$ is a Euclidean distance in R^p .

Unlike the extensive use of the *weight* breakdown point, *local-shift-sensitivity* was never fully exploited. One reason is that, in reality, it is a “global” measure as supremum over all

\mathbf{x}, \mathbf{y} . Thus, λ^* cannot be used to study T 's bias for \mathbf{x} 's small perturbation in the ϵ -mixture, from \mathbf{x} to $\mathbf{x} + \mathbf{h}$, $\|\mathbf{h}\|$ small,

$$T[(1 - \epsilon)F + \epsilon\Delta_{\mathbf{x}+\mathbf{h}}] - T[(1 - \epsilon)F + \epsilon\Delta_{\mathbf{x}}]. \quad (6)$$

Rousseeuw and Leroy (1987) presented a physical analogy to the notion of *weight* breakdown point. A beam is fixed at one end and, at point \mathbf{x} on the beam, a stone with weight ϵ is attached. For small weights, the “deformation” (i.e., the bias) (4) of the beam is linear in ϵ and one can predict the weight's effect. As soon as ϵ takes value larger than the “breakdown value” (that depends on the location \mathbf{x}), (4) cannot be used.

For the physical analogue of *location* breakdown, a sufficiently long beam is used and weight ϵ “travels” at different \mathbf{x} -locations far away from the fixed end of the beam. There is a location $\mathbf{x}_{0,\epsilon}$ that makes the beam “break”. The beam will break also with a small perturbation from $\mathbf{x}_{0,\epsilon} - \mathbf{h}$ to $\mathbf{x}_{0,\epsilon}$, $\|\mathbf{h}\|$ small. This is the reason we study \mathbf{h} -perturbations (6) for remote \mathbf{x} 's.

When F is defined on the real line, to express the physical analogue of *location* breakdown with the derivative of the influence function we evaluate (6) at neighboring points $x, x + h, x \in R, h \in R, |h|$ small.

Lemma 2.1

$$\lim_{h \rightarrow 0} \lim_{\epsilon \rightarrow 0} \frac{T[(1 - \epsilon)F + \epsilon\Delta_{x+h}] - T[(1 - \epsilon)F + \epsilon\Delta_x]}{\epsilon h} = IF'(x; T, F). \quad (7)$$

$IF'(x; T, F)$ is used to approximate (6) for small $\epsilon, |h|$:

$$T[(1 - \epsilon)F + \epsilon\Delta_{x+h}] - T[(1 - \epsilon)F + \epsilon\Delta_x] \approx \epsilon h IF'(x; T, F); \quad (8)$$

(8) is *the tool* used to approximate L_1 and L_2 residuals and determine group influence.

In simple linear L_1 -regression, derivatives of $IF(x, y; T, F)$ are constant where the residual does not vanish; T is any of the regression coefficients. As (x, y) becomes more remote in x , eventually there is a change of the L_1 -regression coefficients at $(x + h, y)$, the L_1 -residual vanishes, derivatives of $IF(x, y; T, F)$ takes values infinities and (8) is *not* valid.

This observation motivates the definition of *location* breakdown point (*LBP*) where the derivative of the influence function takes infinite values. In L_1 and L_2 linear regressions partial derivatives of the coefficients influence functions exist and, in addition, one remote coordinate in the factor space is enough to reach *LBP*. Thus, in the definition of *LBP* for T partial derivatives are used instead of a directional derivative.

Definition 2.2 Let T be a functional defined on probabilities in R^p , with real values, $p \geq 1$. Then, $\mathbf{x} \in R^p$ is Location Breakdown Point (*LBP*) if there is $j \leq p$:

$$|\frac{\partial}{\partial x_j} IF(\mathbf{x}; T, F)| = \infty \text{ or does not exist}; \quad (9)$$

x_j is \mathbf{x} 's j -th coordinate, F is probability.

Example 2.1 Let F be a probability on the real line, $T_1(F)$ is the median of F , $T_2(F)$ is the mean of F and their influence functions are:

$$IF(x; T_1, F) = \frac{\text{sign}[x - T_1(F)]}{2f[T_1(F)]}, \quad IF(x; T_2, F) = x - T_2(F).$$

From (9), there are no *LBP*s on the real line for the mean, T_2 , but for the median, T_1 , its value is the only *LBP*.

Example 2.2 Consider a simple linear regression model, $Y = \beta_0 + \beta_1 X + e$, with error e having mean zero and finite second moment, F is the joint distribution of (X, Y) and $f_{Y|X}$ is the conditional density of Y given X ,

$$\tilde{f}_{Y|X}(x) = f_{Y|X}[\beta_{0,L_1}(F) + \beta_{1,L_1}(F)x|x]. \quad (10)$$

The influence functions for the L_2 -parameters $\beta_{0,L_2}(F)$, $\beta_{1,L_2}(F)$, obtained at F are

$$IF(x, y; \beta_{0,L_2}(F), F) = [y - \beta_{0,L_2}(F) - \beta_{1,L_2}(F)x] \frac{EX^2 - xEX}{Var(X)}, \quad (11)$$

$$IF(x, y; \beta_{1,L_2}(F), F) = [y - \beta_{0,L_2}(F) - \beta_{1,L_2}(F)x] \frac{x - EX}{Var(X)}; \quad (12)$$

EU and $Var(U)$ denote, respectively, U 's mean and variance. The derivatives of influence functions (11), (12) do not satisfy (9) for $x \in R$, $y \in R$, thus there are no *LBP*s.

The influence functions for the L_1 -parameters $\beta_{0,L_1}(F)$, $\beta_{1,L_1}(F)$, obtained at F are

$$IF(x, y; \beta_{0,L_1}(F), F) = \frac{\text{sign}[y - \beta_{0,L_1}(F) - \beta_{1,L_1}(F)x]}{2} \frac{EX^2 \tilde{f}_{Y|X}(X) - xEX \tilde{f}_{Y|X}(X)}{E \tilde{f}_{Y|X}(X) EX^2 \tilde{f}_{Y|X}(X) - (EX \tilde{f}_{Y|X}(X))^2}, \quad (13)$$

$$IF(x, y; \beta_{1,L_1}(F), F) = \frac{\text{sign}[y - \beta_{0,L_1}(F) - \beta_{1,L_1}(F)x]}{2} \frac{xEX \tilde{f}_{Y|X}(X) - EX \tilde{f}_{Y|X}(X)}{E \tilde{f}_{Y|X}(X) EX^2 \tilde{f}_{Y|X}(X) - (EX \tilde{f}_{Y|X}(X))^2}, \quad (14)$$

From (9), *LBP*s in L_1 -regression are all x, y satisfying the relation $y = \beta_{0,L_1}(F) + \beta_{1,L_1}(F)x$.

Remark 2.1 *The y -derivatives of L_2 -influence functions (11), (12) are, respectively, $(EX^2 - xEX)/\text{Var}(X)$ and $(x - EX)/\text{Var}(X)$; those of L_1 -influence functions (13), (14) either vanish or take values at infinities.*

3 Influence, Residuals, Leverage Cases, *RINFIN*

MULTIPLE REGRESSION MODEL

Let (\mathbf{X}, Y) follow probability model F in R^{p+1} ,

$$Y = \mathbf{X}^T \beta + e; \quad (15)$$

$\mathbf{X} = (1, X_1, \dots, X_p)^T$ is the independent variable, Y is the response, $\beta = (\beta_0, \dots, \beta_p)^T$.

The Model Assumptions:

- (A1) The error, e , is symmetric around zero and has finite second moment.
- (A2) X_1, \dots, X_p are independent random variables.
- (A3) Case (\mathbf{x}, y) is mixed with cases from model F with probability ϵ (model $F_{\epsilon, \mathbf{x}, y}$).

Let $(\mathbf{x} + \mathbf{h}, y)$, $(\mathbf{x}, y + h)$ be small perturbations of (\mathbf{x}, y) . The goal is to compare the (\mathbf{x}, y) - residual changes in L_1 and in L_2 regressions:

- i) before (\mathbf{x}, y) enters model F and after, i.e., under $F_{\epsilon, \mathbf{x}, y}$,
- ii) when $(\mathbf{x} + \mathbf{h}, y)$ replaces (\mathbf{x}, y) in the ϵ -mixture, i.e., under $F_{\epsilon, \mathbf{x}, y}$ and $F_{\epsilon, \mathbf{x} + \mathbf{h}, y}$ and
- iii) when $(\mathbf{x}, y + h)$ replaces (\mathbf{x}, y) in the ϵ -mixture, i.e., under $F_{\epsilon, \mathbf{x}, y}$ and $F_{\epsilon, \mathbf{x}, y + h}$.

Let \mathbf{x} become more extreme in the i -th coordinate, $x_i + h$, $|h|$ small; denote by $\mathbf{x}_{i,h}$ this perturbation of \mathbf{x} ,

$$\mathbf{x}_{i,h} = \mathbf{x} + (0, \dots, h, \dots, 0). \quad (16)$$

The j -th regression coefficients obtained by L_m -minimization, respectively, at models $F_{\epsilon, \mathbf{x}, y}$ and F are:

$$\beta_{j, L_m, \mathbf{x}} = \beta_{j, L_m}([F_{\epsilon, \mathbf{x}, y}]), \quad \beta_{j, L_m} = \beta_{j, L_m}([F]), \quad j = 0, 1, \dots, p, \quad (17)$$

$$\beta_{L_m, \mathbf{x}} = (\beta_{0, L_m, \mathbf{x}}, \dots, \beta_{p, L_m, \mathbf{x}})^T, \quad \beta_{L_m} = (\beta_{0, L_m}, \dots, \beta_{p, L_m})^T; \quad (18)$$

denote the L_m -residuals for models $F_{\epsilon, \mathbf{u}, v}$ and F , respectively,

$$r_{m, \mathbf{u}} = r_m(\mathbf{u}, v; F_{\epsilon, \mathbf{u}, v}) = v - \beta_{L_m, \mathbf{u}}^T \mathbf{u}, \quad r_m = r_m(\mathbf{u}, v) = v - \beta_{L_m}^T \mathbf{u}, \quad m = 1, 2. \quad (19)$$

When indices of β 's and r include at least one among $\mathbf{x}, \mathbf{x}_{i,h}, \mathbf{u}, y+h$, they are determined from a gross-error model. Only \mathbf{x} is used at $\beta_{j, L_m, \mathbf{x}}$ and only \mathbf{u} is used at $r_{m, \mathbf{u}}$ because of interest in factor space perturbations and to avoid increasing the number of indices. The influence function of β_{j, L_m} is evaluated at (\mathbf{x}, y) for F , thus use

$$IF_{j, L_m} = IF(\mathbf{x}, y; \beta_{j, L_m}, F), \quad IF'_{v, j, L_m} = \frac{\partial IF(\mathbf{x}, y; \beta_{j, L_m}, F)}{\partial v}, \quad v = y, x_i, \quad (20)$$

i.e., in words, IF'_{v, j, L_m} is the derivative of IF_{j, L_m} with respect to v , $i = 1, \dots, p$, $j = 0, 1, \dots, p$, $m = 1, 2$.

Influence functions of L_m regression coefficients are solutions of the equations:

$$IF_{0, L_m} + IF_{1, L_m} EX_1 + \dots + IF_{p, L_m} EX_p = \tilde{r}_m(\mathbf{x}, y), \quad (21)$$

$$IF_{0, L_m} EX_i + \dots + IF_{p, L_m} EX_i X_j + \dots + IF_{p, L_m} EX_i X_p = x_i \tilde{r}_m(\mathbf{x}, y), \quad i = 1, \dots, p, \quad m = 1, 2, \quad (22)$$

$$\text{with} \quad \tilde{r}_1(\mathbf{x}, y) = \frac{\text{sign}[r_1(\mathbf{x}, y)]}{2\tilde{f}_{Y|\mathbf{X}}}, \quad \tilde{r}_2(\mathbf{x}, y) = r_2(\mathbf{x}, y); \quad (23)$$

from the symmetry of e in assumption (A1), $\tilde{f}_{Y|\mathbf{X}}$ is the common value

$$\tilde{f}_{Y|\mathbf{X}} = f_{Y|\mathbf{X}}(\mathbf{x}) = f_{Y|\mathbf{X}}[\beta_{0, L_1} + \beta_{1, L_1} x_1 + \dots + \beta_{p, L_1} x_p | \mathbf{x}]. \quad (24)$$

E-MATRIX AND ITS COFACTORS

Under assumption (A2), the coefficients in the system of equations (21), (22) form a special type of matrix we call E_p -matrix; p is the covariates' dimension. As an illustration, for real numbers a, b, c, A, B, C ,

$$E_4 = \begin{pmatrix} 1 & a & b & c \\ a & A & ab & ac \\ b & ba & B & bc \\ c & ca & cb & C \end{pmatrix}.$$

For E_4 , the corresponding linear regression model with independent covariates X_1, X_2, X_3 provides $a = EX_1, b = EX_2, c = EX_3$ and $A = EX_1^2, B = EX_2^2, C = EX_3^2$.

Definition 3.1 E_n -matrix with real entries has form:

$$E_n = \begin{pmatrix} 1 & a_1 & a_2 \dots & a_n \\ a_1 & A_1 & a_1 a_2 \dots & a_1 a_n \\ a_2 & a_2 a_1 & A_2 \dots & a_2 a_n \\ \dots & & & \\ a_n & a_n a_1 & a_n a_2 \dots & A_n \end{pmatrix}. \quad (25)$$

Notation: $E_{n,-k}$ denotes the matrix obtained from E_n by deleting its k -th column and k -th row, $2 \leq k \leq n + 1$.

Property of E_n -matrix: Deleting the k -th row and the k -th column of E_n -matrix, the obtained matrix $E_{n,-k}$ is E_{n-1} matrix formed by $\{1, a_1, \dots, a_n\} - \{a_{k-1}\}$, $2 \leq k \leq n + 1$.

The cofactors of E_n -matrix are needed to solve (21), (22).

Proposition 3.1 a) The determinant of E_n -matrix (25) is

$$|E_n| = \prod_{m=1}^n (A_m - a_m^2). \quad (26)$$

b) Let $C_{i+1,j+1}$ be the cofactor of element $(i + 1, j + 1)$ in E_n . Then, its determinant

$$C_{i+1,j+1} = 0, \text{ if } i > 0, j > 0, i \neq j, \quad C_{1,j+1} = -a_j \prod_{k \neq j} (A_k - a_k^2). \quad (27)$$

$$C_{i+1,1} = -a_i \prod_{j \neq i} (A_j - a_j^2), \text{ if } i > 0, \quad C_{1,1} = |E_n| + \sum_{k=1}^n a_k^2 |E_{n,-k}|. \quad (28)$$

Proposition 3.2 *For regression model (15) with assumptions (A1)-(A3), $r_1(\mathbf{x}, y) \neq 0$, and \tilde{r}_m in (23), the influence functions of L_m -regression coefficients, $m = 1, 2$, are:*

$$IF_{0,L_m} = \tilde{r}_m[1-p + \sum_{j=1}^p \frac{EX_j^2 - x_j EX_j}{\sigma_j^2}], \quad IF_{j,L_m} = \tilde{r}_m \frac{x_j - EX_j}{\sigma_j^2}, \quad j = 1, \dots, p; \quad (29)$$

σ_j^2 is the variance of X_j , $j = 1, \dots, p$.

COMPARISON OF L_m -RESIDUALS FOR F , $F_{\epsilon,\mathbf{x},y}$, $F_{\epsilon,\mathbf{x}_{i,h},y}$, $F_{\epsilon,\mathbf{x},y+h}$ $m = 1, 2$

The next proposition confirms that for \mathbf{x} -remote case (\mathbf{x}, y) , the size of L_1 residual is larger than the size of its L_2 residual before (\mathbf{x}, y) reaches L_1 LBP.

Proposition 3.3 *For regression model (15) with (A1)-(A3), perturbation (16) and $r_1(\mathbf{x}, y) \neq 0$:*

a) *For ϵ small:*

a₁) *The difference of (\mathbf{x}, y) -residuals at $F_{\epsilon,\mathbf{x},y}$ and F is:*

$$r_{m,\mathbf{x}}(\mathbf{x}, y) - r_m(\mathbf{x}, y) \approx -\epsilon[IF_{0,L_m} + \sum_{j=1}^p x_j IF_{j,L_m}] = -\epsilon \tilde{r}_m(\mathbf{x}, y)[1 + \sum_{j=1}^p \frac{(x_j - EX_j)^2}{\sigma_j^2}]; \quad (30)$$

$r_{m,\mathbf{x}}(\mathbf{x}, y)$ and $r_m(\mathbf{x}, y)$ have the same sign and $|r_{m,\mathbf{x}}(\mathbf{x}, y)| < |r_m(\mathbf{x}, y)|$, $m = 1, 2$.

a₂) *The ratio:*

$$\frac{r_{2,\mathbf{x}}(\mathbf{x}, y) - r_2(\mathbf{x}, y)}{r_{1,\mathbf{x}}(\mathbf{x}, y) - r_1(\mathbf{x}, y)} \approx 2\tilde{f}_{Y|\mathbf{X}} \frac{r_2(\mathbf{x}, y)}{\text{sign}[r_1(\mathbf{x}, y)]}; \quad (31)$$

$\tilde{f}_{Y|\mathbf{X}}$ is positive constant (24).

b) *For ϵ and $|h|$ both small:*

b₁) *The difference of (\mathbf{x}, y) -residuals at $F_{\epsilon,\mathbf{x},y}$ and $F_{\epsilon,\mathbf{x}_{i,h},y}$ is:*

$$r_{m,\mathbf{x}_{i,h}}(\mathbf{x}_{i,h}, y) - r_{m,\mathbf{x}}(\mathbf{x}, y) + \beta_{i,L_m} h \approx -\epsilon h[IF_{i,L_m} + IF'_{x_i,0,L_m} + \sum_{j=1}^p x_j IF'_{x_i,j,L_m}] - \epsilon h^2 IF'_{x_i,i,L_m}. \quad (32)$$

Thus,

$$r_{1,\mathbf{x}_{i,h}}(\mathbf{x}_{i,h}, y) - r_{1,\mathbf{x}}(\mathbf{x}, y) \approx -\epsilon h \frac{\text{sign}[r_1(\mathbf{x}, y)]}{\tilde{f}_{Y|\mathbf{X}}} \frac{x_i - EX_i}{\sigma_i^2} - \beta_{i,L_1} h - \epsilon h^2 IF'_{x_i,i,L_1}, \quad (33)$$

$$r_{2,\mathbf{x}_{i,h}}(\mathbf{x}_{i,h}, y) - r_{2,\mathbf{x}}(\mathbf{x}, y) \approx -\epsilon h \left\{ 2 \frac{r_2(x_i - EX_i)}{\sigma_i^2} - \beta_{i,L_2} \left[1 + \sum_{j=1}^p \frac{(x_j - EX_j)^2}{\sigma_j^2} \right] \right\} - \beta_{i,L_2} h - \epsilon h^2 IF'_{x_i,i,L_2}. \quad (34)$$

$b_2)$ If, in addition, $|x_i|$ is large,

$$r_{1,\mathbf{x}_{i,h}}(\mathbf{x}_{i,h}, y) - r_{1,\mathbf{x}}(\mathbf{x}, y) \approx -\epsilon h \frac{\text{sign}[r_1(\mathbf{x}, y)] x_i - EX_i}{\tilde{f}_{Y|\mathbf{X}} \sigma_i^2}, \quad (35)$$

$$r_{2,\mathbf{x}_{i,h}}(\mathbf{x}_{i,h}, y) - r_{2,\mathbf{x}}(\mathbf{x}, y) \approx \epsilon h \cdot 3\beta_{i,L_2} \frac{(x_i - EX_i)^2}{\sigma_i^2}, \quad (36)$$

$$\frac{|r_{2,\mathbf{x}_{i,h}}(\mathbf{x}_{i,h}, y) - r_{2,\mathbf{x}}(\mathbf{x}, y)|}{|r_{1,\mathbf{x}_{i,h}}(\mathbf{x}_{i,h}, y) - r_{1,\mathbf{x}}(\mathbf{x}, y)|} \approx 3|\beta_{i,L_2}| \cdot \tilde{f}_{Y|\mathbf{X}} \cdot |x_i - EX_i| \quad (37)$$

c) For ϵ and $|h|$ both small, the difference of (\mathbf{x}, y) -residuals at $F_{\epsilon,\mathbf{x},y+h}$ and $F_{\epsilon,\mathbf{x},y}$ is:

$$r_{2,\mathbf{x},y+h}(\mathbf{x}, y+h) - r_{2,\mathbf{x},y}(\mathbf{x}, y) \approx h - \epsilon h \left[1 + \sum_{j=1}^p \frac{(x_j - EX_j)^2}{\sigma_j^2} \right]. \quad (38)$$

INFLUENCE ON THE DERIVATIVES OF REGRESSION RESIDUALS-RINFIN

Influence is determined from the distance of residuals' derivatives at (\mathbf{x}, y) for model F and gross-error model $F_{\epsilon,\mathbf{x},y}$. The larger the distance is, the larger the influence of (\mathbf{x}, y) is.

\mathbf{x} -Influence on Residuals

For $(\mathbf{x}_{i,h}, y)$ and (\mathbf{x}, y) both under model F ,

$$\frac{r_m(\mathbf{x}_{i,h}, y) - r_m(\mathbf{x}, y)}{h} = -\beta_{i,L_m}, \quad i = 1, \dots, p, \quad m = 1, 2. \quad (39)$$

From the results for gross-error models $F_{\epsilon,\mathbf{x},y}$, $F_{\epsilon,\mathbf{x}_{i,h},y}$, the difference of residuals derivatives is obtained.

Proposition 3.4 For models F , $F_{\epsilon,\mathbf{x},y}$, $F_{\epsilon,\mathbf{x}_{i,h},y}$ and L_m regression it holds

$$\lim_{h \rightarrow 0} \frac{r_{m,\mathbf{x}_{i,h}} - r_{m,\mathbf{x}}}{h} + \beta_{i,L_m} = -\epsilon [IF_{i,L_m} + IF'_{x_i,0,L_m} + \sum_{j=1}^p x_j IF'_{x_i,j,L_m}], \quad i = 1, \dots, p, \quad m = 1, 2. \quad (40)$$

From (39) and (40), the right side of the latter measures influence of \mathbf{x} 's i -th coordinate in the residual's derivative and provides the motivation for defining influence. When $p > 1$, coordinates other than the i -th are involved in $\sum_{j=1}^p x_j IF'_{x_i,j,L_m}$ in (40), motivating the use of two influence indices.

Definition 3.2 For gross-error model $F_{\epsilon, \mathbf{x}, y}$,

a) the influence of \mathbf{x} 's i -th coordinate in the L_m -residual is

$$\epsilon \cdot |IF_{i, L_m}(\mathbf{x}, y) + IF'_{x_i, 0, L_m}(\mathbf{x}, y) + \sum_{j=1}^p x_j IF'_{x_i, j, L_m}(\mathbf{x}, y)|, \quad m = 1, 2, \quad (41)$$

b) the influence of \mathbf{x} in the L_m -residual is

$$\epsilon \cdot \sum_{i=1}^p |IF_{i, L_m}(\mathbf{x}, y) + IF'_{x_i, 0, L_m}(\mathbf{x}, y) + \sum_{j=1}^p x_j IF'_{x_i, j, L_m}(\mathbf{x}, y)|, \quad m = 1, 2. \quad (42)$$

Influences for models $F_{\epsilon_1, \mathbf{x}_1, y_1}$, $F_{\epsilon_2, \mathbf{x}_2, y_2}$ can be compared.

Definition 3.3 Case (\mathbf{x}_1, y_1) with weight ϵ_1 is more influential for L_m -residuals than case (\mathbf{x}_2, y_2) with weight ϵ_2 , $m = 1, 2$, if

$$\begin{aligned} & \epsilon_2 \cdot \sum_{i=1}^p |IF_{i, L_m}(\mathbf{x}_2, y_2) + IF'_{x_i, 0, L_m}(\mathbf{x}_2, y_2) + \sum_{j=1}^p x_{2,j} IF'_{x_i, j, L_m}(\mathbf{x}_2, y_2)| \\ & \leq \epsilon_1 \cdot \sum_{i=1}^p |IF_{i, L_m}(\mathbf{x}_1, y_1) + IF'_{x_i, 0, L_m}(\mathbf{x}_1, y_1) + \sum_{j=1}^p x_{1,j} IF'_{x_i, j, L_m}(\mathbf{x}_1, y_1)|. \end{aligned} \quad (43)$$

The L_2 -Residual Influence Index (*RINFIN*): For gross-error model $F_{\epsilon, \mathbf{x}, y}$, (42) for $m = 2$ becomes from (56) in the Appendix,

$$RINFIN(\mathbf{x}, y; \epsilon, L_2) = \epsilon \cdot \sum_{i=1}^p \left\{ \left| 2 \frac{r_2(\mathbf{x}, y)(x_i - EX_i)}{\sigma_i^2} - \beta_{i, L_2} \left[1 + \sum_{j=1}^p \frac{(x_j - EX_j)^2}{\sigma_j^2} \right] \right| \right\}. \quad (44)$$

Abuse of notation: Using $RINFIN(\mathbf{x}, y)$ instead of $RINFIN(\mathbf{x}, y; \epsilon, L_2)$.

Remote \mathbf{x} 's have large $RINFIN(\mathbf{x}, y; \epsilon, L_2)$.

Proposition 3.5

$$\lim_{|x_i| \rightarrow \infty} RINFIN(\mathbf{x}, y; \epsilon, L_2) = \infty. \quad (45)$$

Remark 3.1 ($RINFIN^*$) To measure strictly the influence of \mathbf{x} 's i -th component, which is dominant when x_i is remote (see (57)), use also:

$$RINFIN^*(\mathbf{x}, y; \epsilon, L_2) = \epsilon \cdot \sum_{i=1}^p \left\{ \left| 2 \frac{r_2(\mathbf{x}, y)(x_i - EX_i)}{\sigma_i^2} - \beta_{i, L_2} \left[1 + \frac{(x_i - EX_i)^2}{\sigma_i^2} \right] \right| \right\}. \quad (46)$$

Note that *RINF*IN dominates *RINF*IN* in the simulations with the normal model in section 4. However *RINF*IN* allows to identify the bad leverage cases in Hawkins-Braduskass data.

y-Influence on Residuals

Since IF'_{y,j,L_1} vanishes for every j , influence index from y -derivatives of residuals is only presented for L_2 -regression.

For $(\mathbf{x}, y + h)$ and (\mathbf{x}, y) both under model F ,

$$\frac{r_2(\mathbf{x}, y + h) - r_2(\mathbf{x}, y)}{h} = 1, \quad i = 1, \dots, p. \quad (47)$$

Proposition 3.6 For models F , $F_{\epsilon, \mathbf{x}, y}$, $F_{\epsilon, \mathbf{x}, y+h}$ and L_2 regression it holds

$$\lim_{h \rightarrow 0} \frac{r_{2, \mathbf{x}, y+h}(\mathbf{x}, y + h) - r_{2, \mathbf{x}, y}(\mathbf{x}, y)}{h} - 1 \approx -\epsilon \left[1 + \sum_{j=1}^p \frac{(x_j - EX_j)^2}{\sigma_j^2} \right]. \quad (48)$$

Remark 3.2 From (48), the y -influence index is

$$\sum_{j=1}^p \frac{(x_j - EX_j)^2}{\sigma_j^2}; \quad (49)$$

it is maximized for cases in the extremes of the \mathbf{x} -coordinates and can be visually implemented with the proposed plot when \mathbf{x} -coordinates have all the same sign.

4 Applications: *RINF*IN and Residuals' Plots

*RINF*IN'S PERFORMANCE IN SIMULATIONS

Data (\mathbf{X}, Y) follows linear model (15) with $\beta = (1.5, .5, 0, 1, 0, 0, 1.5, 0, 0, 0, 1, 0, \dots, 0)$, $p \geq 11$, and with \mathbf{X} obtained from p -dimensional normal distribution, $\mathcal{N}(\mathbf{0}, \Sigma)$, with $\Sigma_{i,j} = .5^{|j-i|}$, $1 \leq i, j \leq p$, as in Alfons, Croux and Gelper (2013, p.11). When $p < 11$, β 's first p coordinates are used and \mathbf{X} is obtained as in the case $p \geq 11$. Contaminated

\mathbf{X} with probability G consists of p independent normal random variables with mean μ and standard deviation 1. Each sample has size $n = 200$ and various values for p and μ are used, $p < n = 200$. Errors e for model F and for G are independent, normal random variables with mean zero and variance 1. The percentage of contaminated cases $\epsilon = .10$. The number of simulated samples $N = 100$.

The 20 contaminated cases are the first cases in the data. $RINFIN$ and $RINFIN^*$ values are calculated and the case numbers of the top 20 values are recorded and compared with contaminated cases 1-20 to calculate the misclassification proportion. The results appear in Table 1. $RINFIN$ misclassification proportion is only reported being uniformly better than $RINFIN^*$ for the data used.

p	$\mu = 0$	$\mu = .5$	$\mu = 1$	$\mu = 1.5$	$\mu = 2$	$\mu = 2.5$	$\mu = 3$
2	0.9095	0.866	0.7895	0.642	0.524	0.3805	0.281
5	0.91	0.857	0.7065	0.5035	0.2935	0.155	0.0685
10	0.9305	0.849	0.621	0.361	0.1365	0.0435	0.0085
15	0.93	0.8485	0.558	0.242	0.062	0.01	0
20	0.9085	0.8425	0.496	0.167	0.0335	0.0015	0
60	0.933	0.755	0.2025	0.0075	0	0	0
100	0.928	0.6815	0.1005	0.001	0	0	0
140	0.9305	0.6275	0.048	0	0	0	0
190	0.9105	0.7025	0.1535	0.0155	0.001	0	5e-04

Table 1: Average misclassification proportion of $RINFIN$ in $N = 100$ simulated samples of size $n=200$, of which 20 are contaminated leverage cases, from p -dimensional, normal-mixtures data with distance of means in each coordinate μ and variance 1.

The simulations follow the spirit in Khan, Van Aelst and Zamar (2007). When there are no contaminated cases, i.e. $\mu = 0$, the misclassification proportion is near 90% for all p -values. For any fixed p -value, the misclassification proportion decreases as μ increases. For any fixed μ -value the misclassification proportion decreases as p increases except for an anomaly when $p = 190$ due to its proximity to the sample size. By increasing n to 250 cases, this anomaly disappears. The *blessing* of high dimensionality is observed; for a

theoretical confirmation see Yatracos (2013, Section 8, Proposition 8.1).

READING RESIDUALS' PLOTS

The goal is to identify quickly cases that do not follow the unknown model F of the data's majority, in particular bad leverage cases. "Naive" plots of *absolute* residuals for L_1 and L_2 regression against the sum of squares of the independent variables are used.

Look for:

- **(A)** remote neighboring plot-points creating visual gaps in the L_1 -plot's residuals but smaller gaps in the L_2 -plot; these are bad leverage cases far from L_1 *LBP*. For a given \mathbf{x} , the gap is "large" when the ratio of absolute residuals from the upper and lower gap's borders is larger or equal to two.
- **(B)** a group of plot-points with *neighboring* horizontal axis projections, distant from the bulk of the plot, with the L_1 -absolute residuals forming a vertical strip and at least one of them near zero; these are bad leverage cases near L_1 *LBP*.
- **(C)** If no unusual leverage cases are identified when plotting against the \mathbf{x} 's square length, plot the absolute residuals against each explanatory variable and check whether there are remote \mathbf{x} -coordinates for which **(A)**, **(B)** hold.
- **(D)** Large absolute residuals, especially at the extremes of the \mathbf{x} -values in the data, indicating bad leverage or other outlying cases.

USING RINFIN WITH DATA

The data

$$D_n = \{(\mathbf{x}_1, y_1), \dots, (\mathbf{x}_n, y_n)\}, \quad D_{n,-m} = D_n - \{(\mathbf{x}_m, y_m)\}.$$

To calculate sample $RINFIN(\mathbf{x}_m, y_m)$ estimate the parameters in (44) and use $\epsilon = 1/n$:

- a) Use $D_{n,-m}$ to obtain L_2 -estimates $\hat{\beta}_{L_2}$ and $\hat{r}_2(\mathbf{x}_m, y_m)$.
- b) Estimate EX_i and σ_i^2 , respectively, by the sample average and sample variance \mathbf{x} -data's i -th coordinate in $D_{n,-m}$, $i = 1, \dots, p$.
- c) Use $\hat{\beta}_{L_2}$'s i -th coordinate and replace x_i with \mathbf{x}_m 's i -th coordinate, $i = 1, \dots, p$.

If a group G of k remote \mathbf{x} -neighboring cases exists,

$$G = \{(\mathbf{x}_{i_1}, y_{i_1}), \dots, (\mathbf{x}_{i_k}, y_{i_k})\} \subset D_n,$$

D_n may follow a gross-error model. Let \bar{g} be the average of the elements in G and use, instead of D_n , new data

$$(D_n - G) \cup \{\bar{g}\}.$$

Calculate *RINF*IN-values following a)-c). For *RINF*IN(\bar{g}) use $\epsilon = k/n$; in the remaining $(n - k)$ cases weights are $1/n$.

With J groups, G_1, \dots, G_J , of remote \mathbf{x} -neighboring cases, $G_k \cap G_l = \emptyset, k \neq l$, obtain averages $\bar{g}_1, \dots, \bar{g}_J$, and use data set

$$(D_n - \cup_{j=1}^J G_j) \cup \{\bar{g}_1, \dots, \bar{g}_J\}.$$

Proceed with a)-c). For *RINF*IN(\bar{g}_j) use $\epsilon_j = k_j/n, k_j$ is the cardinality of $G_j, j = 1, \dots, J$; in the remaining cases weights are $1/n$.

*DATA PLOTS & RINF*IN VALUES

In Figures 1 and 2, L_1 and L_2 plots of absolute regression residuals are presented for twelve, well known data sets; those without reference are in Rousseeuw and Leroy (1987). Several methods fail to determine cases from gross-error component(s). The visual comparison of regression plots, *RINF*IN and *RINF*IN* are informative providing indications.

Six data sets present large visual gaps in L_1 -plots (see (A)) and smaller gaps in L_2 -plots. In one of the former, the Hawkins-Bradru-Kass data, residuals' plots give the impression of more than one gross-error component but *RINF*IN* is successful after grouping. The remaining data sets presenting smaller L_1 -gaps, if any, are: Hertzsprung-Russel, Hadi-Simonoff, Stack Loss, Coleman, Salinity and Modified Wood. The ultimate data set is the most challenging because there are no immediate visual indications, but *RINF*IN is successful without grouping as well as after grouping along with *RINF*IN*.

In Telephone data (covariates' dimension $p = 1$, number of cases $n = 24$, all covariates positive), observations 15-20 cause a large gap in the residuals of the L_1 -plot and no gap

in the L_2 -plot; (D) applies. These are indeed the outliers because of the change in the recording system used.

In Kootenay river data ($p = 1$, $n = 13$, all covariates positive), case 4 is remote and causes a large gap in the L_1 -residuals, unlike the L_2 -residuals. Both (A) and (D) apply. *RINFIN* values confirm the visual findings.

DATA: Kootenay River (p=1, n=13)						
CASE	4	7	2	12	6	1
<i>RINFIN</i>	8.906	0.106	0.052	0.044	0.030	0.015

In Brain and Body data ($p = 1$, $n = 28$, not all covariates positive), cases 6, 16, 25 are remote, obtained from 3 dinosaurs each with small brain and heavy body, and cause a large gap in the L_1 residuals. (A) applies. *RINFIN* values confirm the visual findings. Case 26 in *R* library is case 25 in Rousseeuw and Leroy (1987).

DATA: LogBrain and LogBody (p=1, n=28)						
CASE	25	6	16	27	17	10
<i>RINFIN</i>	0.298	0.183	0.157	0.051	0.039	0.029

In Hertzsprung-Russel star data ($p = 1$, $n = 47$, all covariates positive), cases 11, 20, 30, 34 correspond to giant stars. These are remote, x -neighboring cases and many of the remaining cases have either comparable or larger absolute residuals. Absolute residuals of cases 11, 20, 30, 34 form a narrow strip in the L_1 plot and that of case 11 is near zero, indicating its proximity to L_1 *LBP*. Thus, (B) applies. Barrodale (1968, p. 55, l. -2-p.56, l. 2) observed a similar behavior in an example for cases he called “wild”. *RINFIN* values after grouping x -neighboring cases, $G_1 = \{11, 20, 30, 34\}$, $G_2 = \{7, 14\}$ support that $\{11, 20, 30, 34\}$ are bad leverage cases. Note that after grouping, the cases in the table form an “envelope” in the L_1 and L_2 plots.

DATA: Hertzsprung-Russel stars ($p = 1, n = 47$)					
CASE	$RINFIN$	GROUP	$RINFIN$	GROUP	$RINFIN$
34	0 .545	11,20,30,34	26.555	11,20,30,34	39.654
30	0.387	14	0.276	7,14	0.447
20	0 .272	36	0.131	17	0 .159
14	0 .198	4	0.131	36	0.149
7	0.191	2	0.131	4	0.143
11	0 .162	17	0 .125	2	0.143

In Hawkins-Bradru-Kass (1984) artificial data ($p = 3, n = 75$, all covariates positive), cases 11-14 have the largest absolute residual in the L_1 -plot and are the most distant from cases 15-75. The plot shows three separated, distant groups that could be attributed to two sources of gross errors. Using **(A)**, cases 11-14 are bad leverage cases. Using **(B)**, cases 1-10 are “bad” leverage cases near L_1 LBP . Using two groups, cases 1 – 10 and 11 – 14, $RINFIN^*$ indicates the true “bad” leverage cases 1 – 10. Rousseeuw and van Zomeren (1990, Figure 5) identify cases 1-10 in the plot of standardized LMS residuals against robust distances.

DATA: Hawkins-Bradru-Kass ($p = 3, n = 75$)					
CASE	$RINFIN$	GROUP	$RINFIN$	GROUP	$RINFIN^*$
12	0.523	11-14	17.848	1-10	16.648
14	0.442	1-10	15.983	11-14	14.426
11	0.356	43	0.028	43	0.028
13	0.351	68	0.022	68	0.019
7	0.174	47	0.021	47	0.018
6	0.156	27	0.019	27	0.017
3	0.136	52	0.018	54	0.015
5	0.133	60	0.0170	52	0.015

In Scottish Hill Races data ($p = 2, n = 35$, all covariates positive), cases 7 and 18 both at data’s extremes cause large gap in the L_1 -residuals unlike the L_2 -residuals. **(A)** and **(D)** both apply. According to Atkinson (1986, p. 399) observation 33 is masked by points

7 and 18 and an error is reported for case 18. *RINFIN*-values identify these cases and in addition case 11.

DATA: Scottish Hill Races ($p=2$, $n=35$)							
CASE	7	11	35	33	18	31	17
<i>RINFIN</i>	3.577	3.230	2.492	1.558	1.067	0.796	0.508

In Hadi-Simonoff (1993) data ($p = 2$, $n = 25$, one covariate negative -.116), remote cases 1-3 have the larger absolute residuals in the L_1 plot and case 17 follows. In the L_2 plot case 17 has the largest absolute residual and cases 1-3 have their absolute residuals reduced. (A) applies for cases 1-3. When group $G = \{1, 2, 3\}$ is used, *RINFIN* values indicate these are bad leverage cases. Hadi and Simonoff (1993) identify the *true outliers*, cases 1-3, and report that “clean” cases 6, 11, 13, 17 and 24 have larger absolute Least Median of squares residuals than cases 1-3. Plots of standardized absolute residuals for bounded influence as well as M -estimates regression do not reveal the outliers as unusual cases. Yatracos (2013) identifies cases 1-3 as remote cluster with a projection-pursuit cluster index.

DATA: Hadi-Simonoff ($p = 2$, $n = 25$)			
CASE	<i>RINFIN</i>	GROUP	<i>RINFIN</i>
22	0.677	1,2,3	1.074
4	0.572	4	0.645
17	0.527	17	0.620
12	0.565	22	0.607
25	0.374	12	0.464
1	0.351	13	0.399
2	0.346	25	0.328
3	0.340	24	0.298

In Education data ($p = 3$, $n = 50$, all covariates positive), case 50 (Alaska) causes a large gap in the L_1 -residuals, unlike the L_2 -residuals. (A) applies. *RINFIN* values confirm the visual findings.

DATA: Education ($p = 3, n = 50$)						
CASE	50	33	7	44	29	5
<i>RINFIN</i>	1.20	0.482	0.474	0.407	0.334	0.301

In Stackloss data ($p = 3, n = 21$, all covariates positive) cases 1, 3, 4, 21 form a large gap in the L_1 -plot and the gap is reduced in the L_2 -plot. **(A)** applies. L_1 -plot indicates cases 1 and 3 can form a group with case 2 which has small absolute residual and is near *LBP*. **(B)** applies for cases 1,2,3. *RINFIN** values indicate 1, 2, 3, 21 are bad leverage cases. Case 4 has *RINFIN* and *RINFIN** values before grouping at the 10-th percentile and after grouping below the 40th percentile. Rousseeuw and van Zomeren (1990) and Flores (2015) identify cases 1, 2, 3, 4, 21 as bad leverage cases using, respectively, plots of standardized LMS residuals against robust distances and leverage constants.

DATA: Stackloss ($p = 3, n = 21$)							
CASE	<i>RINFIN</i>	CASE	<i>RINFIN</i> *	GROUP	<i>RINFIN</i>	GROUP	<i>RINFIN</i> *
17	1.696	2	0.885	1,2,3	1.664	1,2,3	1.779
2	1.527	12	0.428	17	1.481	21	0.565
1	0.757	21	0.427	21	0.697	12	0.444
15	0.557	17	0.420	7	0.642	7	0.409
12	0.524	15	0.380	15	0.535	17	0.368
18	0.520	11	0.317	8	0.531	15	0.358
7	0.519	7	0.315	12	0.528	11	0.308
8	0.440	16	0.264	18	0.455	8	0.301

In Coleman data ($p = 5, n = 20$, not all covariates positive) cases 3 and 18 cause a large gap in the L_1 -plot and **(D)** applies. Case 18 has larger absolute residual than most of the remaining cases and lives at the \mathbf{x} -extremes. Cases 3, 4, 9, 16 in the L_1 -plot indicate a potential cluster near *LBP*. According to Rousseeuw and Leroy (1987) “.... examining the Least Squares results, ... cases 3, 11 and 18 are furthest away from the linear model. ... The robust regression spots schools 3, 17 and 18 as outliers ...”. *RINFIN* and *RINFIN** highest four values identify cases 1,6,11, 18, 19.

DATA: Coleman (p=5, n=20)								
CASE	6	11	1	19	10	15	2	16
<i>RINFIN</i>	4.125	3.943	3.834	3.627	2.644	2.548	2.439	1.984
CASE	11	1	19	18	3	12	6	16
<i>RINFIN*</i>	2.297	1.736	1.316	1.291	1.279	1.269	1.150	1.076

In Salinity data (Ruppert and Carroll, 1980, $p = 3$, $n = 28$, all covariates positive) case 16 has the largest absolute residual, is \mathbf{x} -remote and the gap caused in the L_1 plot is small. In the L_2 plot its absolute residual is reduced. Both (A) and (D) apply for case 16. *RINFIN* values confirm the visual findings. In Carroll and Ruppert (1985) the analysis of the data shows that cases 3 and 16 are masking case 5.

DATA: Salinity (p=3, n=28)						
CASE	16	15	5	3	9	4
<i>RINFIN</i>	2.216	0.418	0.327	0.307	0.293	0.288

In modified Wood data ($p = 5$, $n = 20$, all covariates positive) there is no visual gap in the L_1 -plot. Since covariates are positive, cases 7, 19, 1, 4, 6, 8 live at one \mathbf{x} -extreme of the data and present a pattern like that described in (B), with the residual of case 8 near 0. To determine the neighboring cases, plot L_2 (or L_1) residuals against each \mathbf{x} -coordinate. In Figure 3 it is clear that cases 4, 6, 8, 19 are neighboring and remote in each coordinate. Cases 4, 6, 8, 19 form also a strip in the last two L_2 -plots in Figure 3. (B) applies for these cases in view of the L_1 plot in Figure 1. This is confirmed by the four higher *RINFIN* values and also by both *RINFIN* and *RINFIN** values when $\{4, 6, 8, 19\}$ are considered as group.

DATA: Modified Wood ($p = 5, n = 20$)							
CASE	<i>RINFIN</i>	CASE	<i>RINFIN*</i>	GROUP	<i>RINFIN</i>	GROUP	<i>RINFIN*</i>
19	1.579	11	0.871	4,6,8,19	34.729	4,6,8,19	29.583
8	1.532	12	0.508	11	1.710	11	1.366
6	1.452	1	0.493	7	1.460	7	0.597
4	1.3312	7	0.476	12	1.390	12	0.556
12	1.324	14	0.448	10	1.084	1	0.468
11	1.161	19	0.442	16	0.785	14	0.389
7	1.158	8	0.434	1	0.779	16	0.285
10	1.075	4	0.386	17	0.738	10	0.252

5 APPENDIX

Proof of Lemma 2.1: Equality (7) is obtained by adding and subtracting $T(F)$ in the numerator of its left side and by taking first the limit with respect to ϵ . \square

Proof for Proposition 3.1: a) Induction is used.

For $n = 1$, the determinant is $A_1 - a_1^2$.

For $n = 2$, the determinant is

$$\begin{aligned}
(A_1 A_2 - a_1^2 a_2^2) - a_1 \cdot (a_1 A_2 - a_1 a_2^2) + a_2 \cdot (a_1^2 a_2 - A_1 a_2) &= A_1 A_2 - a_1^2 A_2 + a_1^2 a_2^2 - A_1 a_2^2 \\
&= A_2(A_1 - a_1^2) - a_2^2(A_1 - a_1^2) = (A_1 - a_1^2)(A_2 - a_2^2).
\end{aligned}$$

Assume that (26) holds for E_n . To show it holds for E_{n+1} consider the matrix E_{n+1} :

$$E_{n+1} = \begin{pmatrix} 1 & a_1 & a_2 \dots & a_n & a_{n+1} \\ a_1 & A_1 & a_1 a_2 \dots & a_1 a_n & a_1 a_{n+1} \\ a_2 & a_2 a_1 & A_2 \dots & a_2 a_n & a_2 a_{n+1} \\ \dots & & & & \\ a_n & a_n a_1 & a_n a_2 \dots & A_n & a_n a_{n+1} \\ a_{n+1} & a_{n+1} a_1 & a_{n+1} a_2 \dots & a_{n+1} a_n & A_{n+1} \end{pmatrix}$$

$|E_{n+1}|$ is obtained using line $(n+1)$ and its cofactors $C_{n+1,1}, \dots, C_{n+1,n+1}$:

$$|E_{n+1}| = a_{n+1} C_{n+1,1} + a_{n+1} a_1 C_{n+1,2} + \dots + a_{n+1} a_n C_{n+1,n} + A_{n+1} C_{n+1,n+1}. \quad (50)$$

Observe that for $2 \leq j \leq n$, cofactor $C_{n+1,j}$ is obtained from a matrix where the last column is a multiple of its first column by a_{n+1} , thus,

$$C_{n+1,j} = 0, \quad j = 2, \dots, n. \quad (51)$$

For the matrix in cofactor $C_{n+1,1}$, observe that in its last column a_{n+1} is common factor and if taken out of the determinant the remaining column is the vector generating E_n , i.e. $\{1, a_1, \dots, a_n\}$. With $n - 1$ successive interchanges to the left, this column becomes first and E_n appears. Thus,

$$C_{n+1,1} = (-1)^{n+2}(-1)^{n-1} \cdot a_{n+1}|E_n| = -a_{n+1}|E_n|. \quad (52)$$

In cofactor $C_{n+1,n+1}$, the determinant is that of E_n ,

$$C_{n+1,n+1} = (-1)^{2(n+1)}|E_n| = |E_n|. \quad (53)$$

From (50)-(53) it follows that

$$|E_{n+1}| = -a_{n+1}^2|E_n| + A_{n+1}|E_n| = \Pi_{m=1}^{n+1}(A_m - a_m^2).$$

b) We now work with E_n . For $i > 0, j > 0, i \neq j$, after deleting row $(j + 1)$ the remaining of column $(j + 1)$ in the cofactor is a multiple of column 1, thus $|C_{i+1,j+1}|$ vanishes.

For $C_{1,j+1}$, using column $j + 1$ to calculate E_n , it holds:

$$a_j C_{1,j+1} + A_j C_{j+1,j+1} = |E_n| \rightarrow a_j C_{1,j+1} = -a_j^2 \Pi_{k \neq j} (A_k - a_k^2) \rightarrow C_{1,j+1} = -a_j \Pi_{k \neq j} (A_k - a_k^2).$$

For $C_{i+1,1}, i > 0$, after deletion of row $(i + 1)$ in E_n the remaining of column $(i + 1)$ in the cofactor's matrix is multiple of a_i and the basic vector creating $E_{n,-i}$. Column 1 of E_n is also deleted and for column $(i + 1)$ in the cofactor's matrix to become first column $(i - 1)$ exchanges of columns are needed. Thus,

$$C_{i+1,1} = (-1)^{i+2} \cdot a_i \cdot (-1)^{i-1} \Pi_{k \neq i} (A_k - a_k^2) = -a_i \cdot \Pi_{k \neq i} (A_k - a_k^2).$$

For $C_{1,1}$ we express $|E_n|$ as sum of cofactors along the first row of E_n ,

$$C_{1,1} + a_1 C_{1,2} + \dots + a_n C_{1,n} = |E_n|$$

$$\rightarrow C_{1,1} = \Pi_{k=1}^n (A_k - a_k^2) + a_1^2 \Pi_{k \neq 1} (A_k - a_k^2) + \dots + a_n^2 \Pi_{k \neq n} (A_k - a_k^2). \quad \square$$

Proof of Proposition 3.2: For system of equations (21), (22) and matrix E_p with $a_j = EX_j$, $A_j = EX_j^2$, $j = 1, \dots, p$, from Proposition 3.1

$$IF_{j,L_m} = \frac{C_{1,j+1}\tilde{r}_m + C_{j+1,j+1}\tilde{r}_m x_j}{|E_p|} = \tilde{r}_m \frac{-EX_j \Pi_{k \neq j} \sigma_k^2 + x_j \Pi_{k \neq j} \sigma_k^2}{\Pi_{k=1}^p \sigma_k^2} = \tilde{r}_m \frac{x_j - EX_j}{\sigma_j^2}, \quad j = 1, \dots, p.$$

$$\begin{aligned} IF_{0,L_m} &= \frac{C_{1,1}\tilde{r}_m + \sum_{j=1}^p C_{1,j+1}\tilde{r}_m x_j}{|E_p|} = \tilde{r}_m \frac{\Pi_{k=1}^p \sigma_k^2 + \sum_{j=1}^p (EX_j)^2 \Pi_{k \neq j} \sigma_k^2 - \sum_{j=1}^p x_j EX_j \Pi_{k \neq j} \sigma_k^2}{\Pi_{k=1}^p \sigma_k^2} \\ &= \tilde{r}_m \left[1 + \sum_{j=1}^p \frac{EX_j^2 - \sigma_j^2 - x_j EX_j}{\sigma_j^2} \right] = \tilde{r}_m \left[1 - p + \sum_{j=1}^p \frac{EX_j^2 - x_j EX_j}{\sigma_j^2} \right]. \quad \square \end{aligned}$$

Lemma 5.1 For the influence functions (29) it holds:

a)

$$IF_{0,L_m} + \sum_{j=1}^p x_j IF_{j,L_m} = \tilde{r}_m \left[1 + \sum_{j=1}^p \frac{(x_j - EX_j)^2}{\sigma_j^2} \right], \quad m = 1, 2, \quad (54)$$

b)

$$IF_{i,L_1} + IF'_{x_i,0,L_1} + \sum_{j=1}^p x_j IF'_{x_i,j,L_1} = \frac{\text{sign}[r_1(\mathbf{x}, y)] x_i - EX_i}{\tilde{f}_{Y|\mathbf{X}}} \frac{1}{\sigma_i^2}, \quad (55)$$

c)

$$IF_{i,L_2} + IF'_{x_i,0,L_2} + \sum_{j=1}^p x_j IF'_{x_i,j,L_2} = 2 \frac{r_2(\mathbf{x}, y)(x_i - EX_i)}{\sigma_i^2} - \beta_{i,L_2} \left[1 + \sum_{j=1}^p \frac{(x_j - EX_j)^2}{\sigma_j^2} \right] \quad (56)$$

$$\approx -3\beta_{i,L_2} \frac{(x_i - EX_i)^2}{\sigma_i^2}, \quad \text{if } |x_i - EX_i| \text{ is very large,} \quad (57)$$

d)

$$IF'_{y,0,L_2} + \sum_{j=1}^p x_j IF'_{y,j,L_2} = 1 + \sum_{j=1}^p \frac{(x_j - EX_j)^2}{\sigma_j^2}. \quad (58)$$

Proof of Lemma 5.1: a) From (29),

$$\begin{aligned} IF_{0,L_m} + \sum_{j=1}^p x_j IF_{j,L_m} &= \tilde{r}_m \left[1 - p + \sum_{j=1}^p \frac{EX_j^2 - x_j EX_j}{\sigma_j^2} \right] + \sum_{j=1}^p x_j \frac{\tilde{r}_m (x_j - EX_j)}{\sigma_j^2} \\ &= \tilde{r}_m \left[1 - p + \sum_{j=1}^p \frac{EX_j^2 - 2x_j EX_j + x_j^2}{\sigma_j^2} \right] = \tilde{r}_m \left[1 + \sum_{j=1}^p \frac{(x_j - EX_j)^2}{\sigma_j^2} \right], \quad m = 1, 2. \end{aligned}$$

b) Proof is provided for $i = 1$. If the residual of (\mathbf{x}, y) does not vanish, since

$$IF_{0,L_1} = \frac{\text{sign}[r_1(\mathbf{x}, y)]}{2\tilde{f}_{Y|\mathbf{X}}} [1 - p + \sum_{j=1}^p \frac{EX_j^2 - x_j EX_j}{\sigma_j^2}], \quad IF_{j,L_1} = \frac{\text{sign}[r_1(\mathbf{x}, y)]}{2\tilde{f}_{Y|\mathbf{X}}} \frac{x_j - EX_j}{\sigma_j^2}, \quad j = 1, \dots, p,$$

$$IF'_{x_1,0,L_1} = -\frac{\text{sign}[r_1(\mathbf{x}, y)]}{2\tilde{f}_{Y|\mathbf{X}}} \frac{EX_1}{\sigma_1^2}$$

$$IF'_{x_1,1,L_1} = \frac{\text{sign}[r_1(\mathbf{x}, y)]}{2\tilde{f}_{Y|\mathbf{X}}\sigma_1^2}, \quad IF'_{x_1,j,L_1} = 0, \quad j \neq 1.$$

Thus,

$$\begin{aligned} & IF_{1,L_1} + IF'_{x_1,0,L_1} + x_1 IF'_{x_1,1,L_1} + x_2 IF'_{x_1,2,L_1} + \dots + x_p IF'_{x_1,p,L_1} \\ &= \frac{\text{sign}[r_1(\mathbf{x}, y)]}{2\tilde{f}_{Y|\mathbf{X}}} \frac{x_1 - EX_1}{\sigma_1^2} - \frac{\text{sign}[r_1(\mathbf{x}, y)]}{2\tilde{f}_{Y|\mathbf{X}}} \frac{EX_1}{\sigma_1^2} + x_1 \frac{\text{sign}[r_1(\mathbf{x}, y)]}{2\tilde{f}_{Y|\mathbf{X}}\sigma_1^2} = \frac{\text{sign}[r_1(\mathbf{x}, y)]}{\tilde{f}_{Y|\mathbf{X}}} \frac{x_1 - EX_1}{\sigma_1^2} \end{aligned}$$

c) Proof is provided for $i = 1$. Since

$$IF_{0,L_2} = r_2 [1 - p + \sum_{j=1}^p \frac{EX_j^2 - x_j EX_j}{\sigma_j^2}], \quad IF_{j,L_2} = r_2 \frac{x_j - EX_j}{\sigma_j^2}, \quad j = 1, \dots, p,$$

$$IF'_{x_1,0,L_2} = -\beta_{1,L_2} [1 - p + \sum_{j=1}^p \frac{EX_j^2 - x_j EX_j}{\sigma_j^2}] - r_2 \frac{EX_1}{\sigma_1^2}$$

$$IF'_{x_1,1,L_2} = -\beta_{1,L_2} \frac{x_1 - EX_1}{\sigma_1^2} + \frac{r_2}{\sigma_1^2} \rightarrow x_1 IF'_{x_1,1,L_2} = -\beta_{1,L_2} \frac{x_1^2 - x_1 EX_1}{\sigma_1^2} + r_2 \frac{x_1}{\sigma_1^2}$$

$$IF'_{x_1,j,L_2} = -\beta_{1,L_2} \frac{x_j - EX_j}{\sigma_j^2} \rightarrow x_j IF'_{x_1,j,L_2} = -\beta_{1,L_2} \frac{x_j^2 - x_j EX_j}{\sigma_j^2}, \quad j \neq 1.$$

Thus,

$$\begin{aligned} & IF_{1,L_2} + IF'_{x_1,0,L_2} + x_1 IF'_{x_1,1,L_2} + x_2 IF'_{x_1,2,L_2} + \dots + x_p IF'_{x_1,p,L_2} \\ &= 2 \frac{r_2(x_1 - EX_1)}{\sigma_1^2} - \beta_{1,L_2} [1 - p + \sum_{j=1}^p \frac{x_j^2 - 2x_j EX_j + EX_j^2}{\sigma_j^2}] \\ &= 2 \frac{r_2(x_1 - EX_1)}{\sigma_1^2} - \beta_{1,L_2} [1 + \sum_{j=1}^p \frac{(x_j - EX_j)^2}{\sigma_j^2}]. \end{aligned}$$

Since

$$\begin{aligned} & r_2(x_1 - EX_1) = y(x_1 - EX_1) - \beta_{1,L_2} x_1(x_1 - EX_1) - (x_1 - EX_1) \sum_{j=2}^p \beta_{j,L_2} x_j \\ &= y(x_1 - EX_1) - \beta_{1,L_2} (x_1 - EX_1)^2 - \beta_{1,L_2} EX_1(x_1 - EX_1) - (x_1 - EX_1) \sum_{j=2}^p \beta_{j,L_2} x_j, \end{aligned}$$

if $|x_1 - EX_1|$ is very large dominating all the other terms, then

$$IF_{1,L_2} + IF'_{x_1,0,L_2} + x_1 IF'_{x_1,1,L_2} + x_2 IF'_{2,x_1,L_2} + \dots + x_p IF'_{p,x_1,L_2} \approx -3\beta_{1,L_2} \frac{(x_1 - EX_1)^2}{\sigma_1^2}.$$

d) From (29),

$$IF'_{y,0,L_2} = 1 - p + \sum_{j=1}^p \frac{EX_j^2 - x_j EX_j}{\sigma_j^2}, \quad IF'_{y,j,L_2} = \frac{x_j - EX_j}{\sigma_j^2}, \quad j = 1, \dots, p.$$

Thus,

$$IF'_{y,0,L_2} + \sum_{j=1}^p x_j IF'_{y,j,L_2} = 1 - p + \sum_{j=1}^p \frac{EX_j^2 - x_j EX_j + x_j^2 - x_j EX_j}{\sigma_j^2} = 1 + \sum_{j=1}^p \frac{(x_j - EX_j)^2}{\sigma_j^2}. \quad \square$$

A Lemma used repeatedly to calculate residuals' differences is due to (4), (8).

Lemma 5.2 *For regression model (15) with assumptions (A1), (A3), perturbation (16), $r_1(\mathbf{x}, y) \neq 0$, and $\epsilon, |h|$ both small:*

$$\beta_{j,L_m,\mathbf{x}} \approx \beta_{j,L_m} + \epsilon IF_{j,L_m}, \quad \beta_{j,L_m,\mathbf{x}_{i,h}} \approx \beta_{j,L_m,\mathbf{x}} + \epsilon h IF'_{x_i,j,L_m}. \quad (59)$$

Proof of Lemma 5.2: Use approximations (4), (8). \square

Proof of Proposition 3.3: a) For a_1), from Lemma 5.2,

$$\begin{aligned} r_{m,\mathbf{x}} &= y - \beta_{0,L_m,\mathbf{x}} - \beta_{1,L_m,\mathbf{x}} x_1 - \dots - \beta_{p,L_m,\mathbf{x}} x_p \\ &\approx y - (\beta_{0,L_m} + \epsilon IF_{0,L_m}) - (\beta_{1,L_m} + \epsilon IF_{1,L_m}) x_1 - \dots - (\beta_{p,L_m} + \epsilon IF_{p,L_m}) x_p \\ &= r_m - \epsilon (IF_{0,L_m} + x_1 IF_{1,L_m} + \dots + x_p IF_{p,L_m}). \end{aligned}$$

(30) follows from (54). Since $\tilde{r}_m(\mathbf{x}, y)$ has the same sign with $r_m(\mathbf{x}, y)$, for ϵ small $r_{m,\mathbf{x}}(\mathbf{x}, y)$ will also have the same sign and reduced size because $-\epsilon \tilde{r}_m(\mathbf{x}, y)$ has opposite sign from $r_m(\mathbf{x}, y)$.

For a_2), (31) follows from (23).

b) Provided for $i = 1$ using Lemma 5.2:

$$\begin{aligned} r_{m,\mathbf{x}_{1,h}} &= y - \beta_{0,L_m,\mathbf{x}_{1,h}} - \beta_{1,L_m,\mathbf{x}_{1,h}} (x_1 + h) - \dots - \beta_{p,L_m,\mathbf{x}_{1,h}} x_p \\ &\approx y - [\beta_{0,L_m,\mathbf{x}} + \epsilon h IF'_{x_1,0,L_m}] - [\beta_{1,L_m,\mathbf{x}} + \epsilon h IF'_{x_1,1,L_m}] (x_1 + h) - \dots - [\beta_{p,L_m,\mathbf{x}} + \epsilon h IF'_{x_1,p,L_m}] x_p \end{aligned}$$

$$\begin{aligned}
&= r_{m,\mathbf{x}} - \beta_{1,L_m,\mathbf{x}} h - \epsilon h [IF'_{x_1,0,L_m} + x_1 IF'_{x_1,1,L_m} + x_2 IF'_{x_1,2,L_m} + \dots + x_p IF'_{x_1,p,L_m}] - \epsilon h^2 IF'_{x_1,1,L_m} \\
&= r_{m,\mathbf{x}} - \beta_{1,L_m} h - \epsilon h [IF'_{i,L_m} + IF'_{x_1,0,L_m} + x_1 IF'_{x_1,1,L_m} + x_2 IF'_{x_1,2,L_m} + \dots + x_p IF'_{x_1,p,L_m}] - \epsilon h^2 IF'_{x_1,1,L_m}.
\end{aligned}$$

(33), (34) follow from (55), (56).

For b_2), if $|x_i|$ is large and $|h|$ is small, $\beta_{i,L_m} h$ and $\epsilon h^2 IF'_{x_i,i,L_m}$ are of smaller order than the remaining terms and (35) follows, in addition, (57) implies (36) and (37) follows also.

$$\begin{aligned}
c) \quad & r_{2,\mathbf{x},y+h}(\mathbf{x}, y+h) = y+h - \beta_{0,L_2,\mathbf{x},y+h} - \beta_{1,L_2,\mathbf{x},y+h} x_1 - \dots - \beta_{p,L_2,\mathbf{x},y+h} x_p \\
& \approx y+h - [\beta_{0,L_2,\mathbf{x},y} + \epsilon h IF'_{y,0,L_2}] - [\beta_{1,L_2,\mathbf{x},y} + \epsilon h IF'_{y,1,L_2}] x_1 - \dots - [\beta_{p,L_2,\mathbf{x},y} + \epsilon h IF'_{y,p,L_2}] x_p \\
& = r_{2,\mathbf{x},y}(\mathbf{x}, y) + h - \epsilon h [IF'_{y,0,L_2} + \sum_{j=1}^p x_j IF'_{y,j,L_2}] = r_{2,\mathbf{x},y}(\mathbf{x}, y) + h - \epsilon h [1 + \sum_{j=1}^p \frac{(x_j - EX_j)^2}{\sigma_j^2}],
\end{aligned}$$

with the last equality obtained from (58). \square

Proof of Proposition 3.4: Follows from (32) dividing both its sides by h and taking the limit with h converging to zero. \square

Proof of Proposition 3.5:

$$\begin{aligned}
\lim_{|x_i| \rightarrow \infty} RINFIN(\mathbf{x}, y; \epsilon, L_2) &\geq \epsilon \cdot \lim_{|x_i| \rightarrow \infty} \left| 2 \frac{r_2(\mathbf{x}, y)(x_i - EX_i)}{\sigma_i^2} - \beta_{i,L_2} \left[1 + \sum_{j=1}^p \frac{(x_j - EX_j)^2}{\sigma_j^2} \right] \right| \\
&\approx \lim_{|x_i| \rightarrow \infty} 3\beta_{i,L_2} \frac{(x_i - EX_i)^2}{\sigma_i^2} = \infty;
\end{aligned}$$

last approximation follows from (57). \square

Proof of Proposition 3.6: Follows from (38) dividing both its sides by h and taking the limit with h converging to zero. \square

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NAIVE L1-RESIDUAL PLOTS

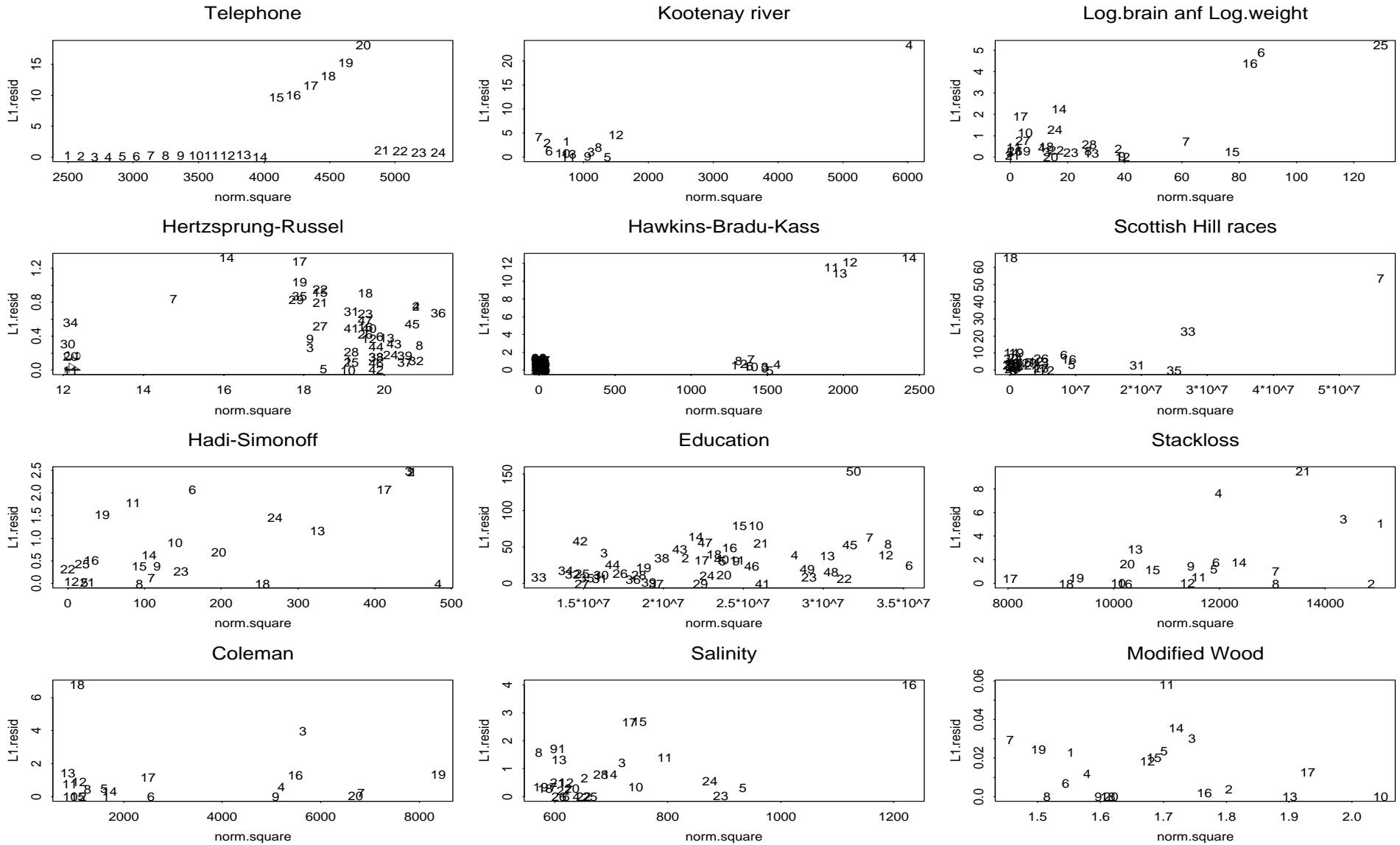


Figure 1

NAIVE L2-RESIDUAL PLOTS

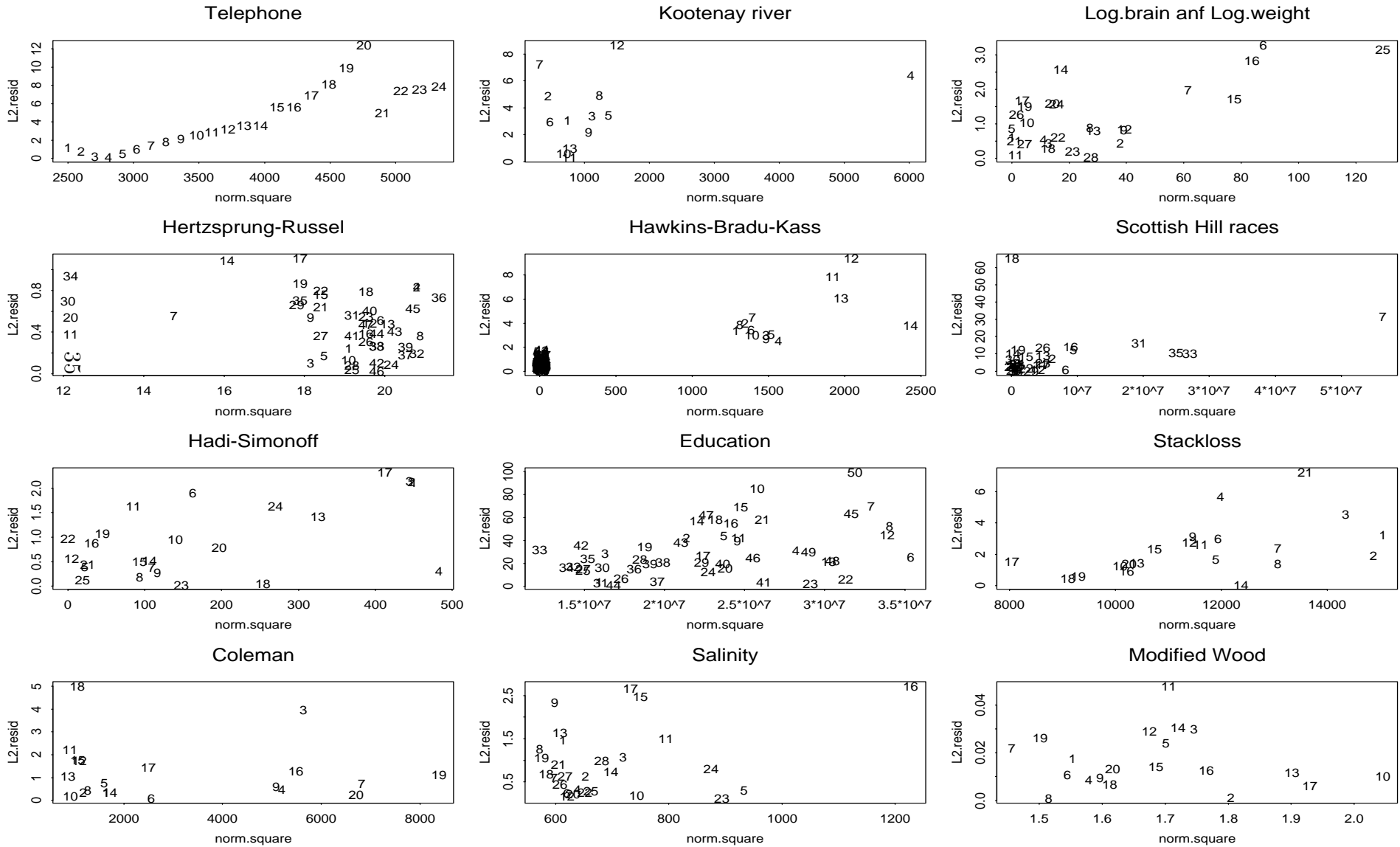


Figure 2

MODIFIED WOOD DATA

ABS. RESIDUALS AGAINST INDEP. VARIABLES

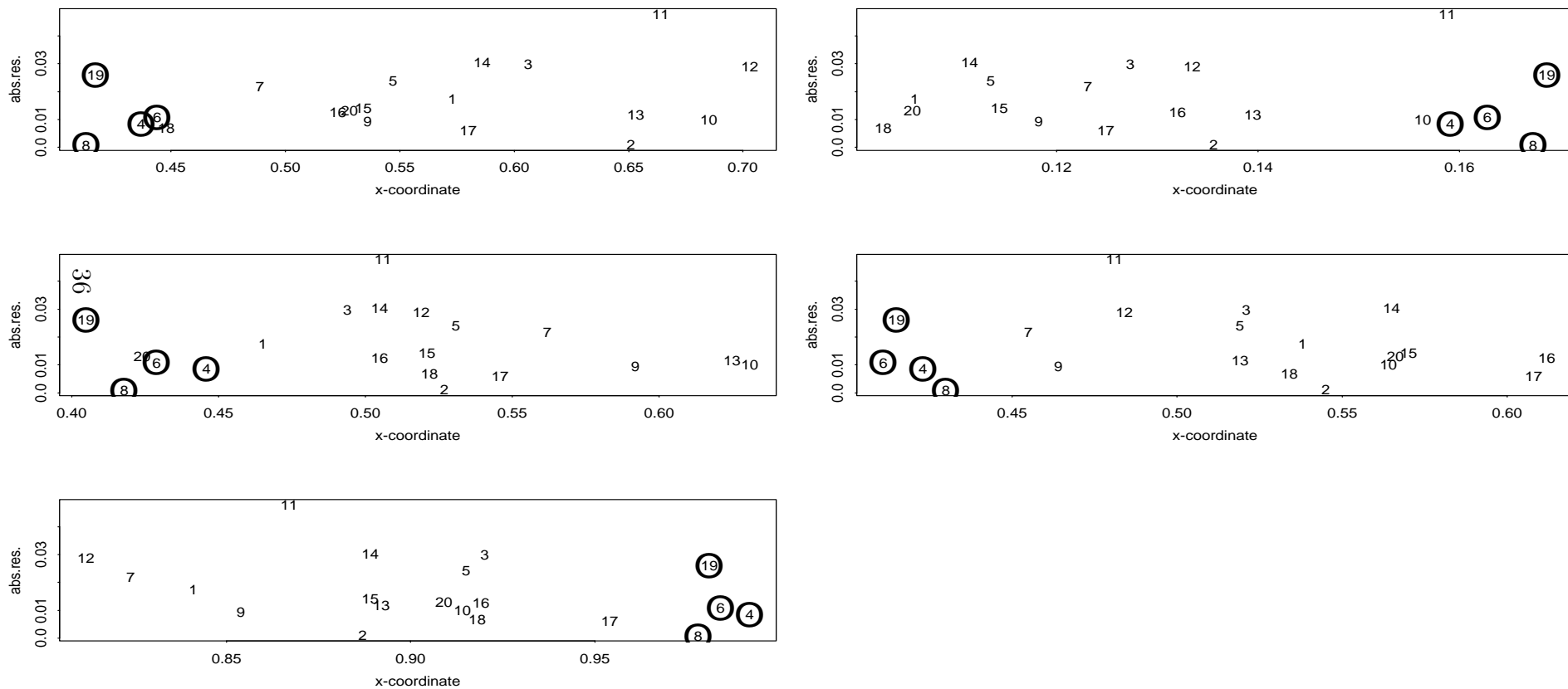


Figure 3

Observations 4, 6, 8, 19 are extreme in all the x-coordinates 1-5